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Within a critical strip there cannot be more than one curve $C(p, q)$, since no two such curves have the same parameter. On the other hand, since the numbers $q - \mu p$ form an everywhere dense set, there are infinitely many curves $C(p, q)$ between any two characteristics not having the same parameter; and so the curves $C(p, q)$ form a set which is dense everywhere in the plane, except within the critical strips if there are any.

6. To a given pair of values for x, τ, there corresponds one, and (unless the value for τ is critical) only one, value for y. y is thus defined as a function of x, τ; and, $\tau(x, y)$ being continuous in $x\ y$, it follows that y is a continuous function of x, τ for all those values of x, τ which make it single-valued.

If, therefore, $\tau(x, y)$ has no critical values, $y(x, \tau)$ is always single-valued and continuous. Let us consider the transformation T given by

$$X = x, \quad Y = \tau(x, y) + \mu x;$$

and its inverse $\qquad x = X: \quad y = y(X, Y - \mu X).$

It is a homeomorphic (*i.e.*, one-to-one and continuous) transformation between (x, y) and (X, Y). Moreover, if we replace (x, y) by $(x + p, y + q)$, (X, Y) is replaced by $(X + p, Y + q)$; and the converse is also true, as T is a one-to-one transformation. T transforms the characteristics in the plane (x, y) into parallel straight lines

$$Y = \mu X + \text{constant}.$$

It is easy to prove* that, if μ is irrational and if $\phi(X, Y)$ is a continuous function, periodic both in X and in Y with the period 1, the average of ϕ along a straight line $Y = \mu X + \text{constant}$ is equal to its average within the fundamental square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$:

$$\lim_{x \to \infty} \frac{1}{x - x_0} \int_{x_0}^{x} \phi(x, y)\, dx = \int_0^1 \int_0^1 \phi(x, y)\, dx\, dy$$

---

* See for instance Weyl, "*Ueber die Gleichverteilung von Zahlen mod. Eins*," *Math. Ann.*, t. 77 (1917)

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the integral on the left-hand side being taken along $Y = \mu X +$ constant. It follows that if $\phi(x, y)$ is continuous and periodic both in x and in y with the period 1, then its average along a characteristic can be expressed by the formula

$$\text{(H)} \qquad \lim_{x \to \infty} \frac{1}{x - x_0} \int_{x_0}^{x} \phi(x, y)\, dx = \int_0^1 dx \left[\int_0^1 \phi(x, y)\, d_y\, \tau(x, y) \right]$$

where the integral on the left-hand side is taken along a given characteristic, and the integral within brackets on the right-hand side is a Stieltjes integral taken for x = constant.

This formula is the expression, in this particular problem, of what is known as the "quasi-ergodic hypothesis."

7. If $\tau(x, y)$ has critical values, the transformation T ceases to be homeomorphic. Nevertheless, it can still be applied successfully to the study of the characteristics, and in particular to the proof of formula (H), which remains true even in this case, although here the "quasi-ergodic hypothesis" ceases to be true in its classical sense since the Stieltjes integral now vanishes identically on all the intervals where τ remains constant. This and other similar questions we shall leave to the reader.

SOME DEFINITE INTEGRALS INVOLVING THE PARABOLIC CYLINDER FUNCTIONS.

By S. C. Mitra, Ph.D.

1. *Let

$$\int_a^b f(x)\,\mathrm{F}(nx)\,dx = \psi(n),$$

and

$$\int_\alpha^\beta \phi(x)\,\mathrm{F}(nx)\,dx = \chi(n).$$

Suppose the functions f, ϕ and F are such that the order of integration can be changed. Then we have

$$\int_a^b f(x)\,\chi(nx)\,dx = \int_\alpha^\beta \phi(y)\,dy \int_a^b f(x)\,\mathrm{F}(nxy)\,dx$$

$$= \int_\alpha^\beta \phi(y)\,\psi(ny)\,dy \qquad \ldots \quad \text{(A)}$$

Many interesting relations between definite integrals may be deduced from this result. We know that if k is real

$$\int_0^\infty \frac{\cos 2kx}{\cosh \pi x}\,dx = \frac{1}{2\cosh k},$$

$$\int_0^\infty \frac{\cos 2kx}{1 + 2\cosh \frac{2}{3}\pi x}\,dx = \frac{\sqrt{3}}{2(1 + 2\cosh 2k)},$$

$$\int_0^\infty \frac{\sin kx}{\sinh \pi x}\,dx = \tfrac{1}{2}\tanh \tfrac{1}{2}k,$$

---

* This follows from Parseval's Theorem, see *Jour. L. M. S.*, January, 1931. Also *Jour. L. M. S.*, Vol. 3, p. 278.

$$\int_0^\infty \frac{\cosh \frac{1}{2} \pi x}{\cosh \pi x} \cos kx dx = \frac{1}{\sqrt{2}} \frac{\cosh \frac{1}{2} k}{\cosh k},$$

$$^* \int_0^\infty D_{2n}(x) \cos kx dx = (-1)^n \sqrt{\pi}\, D_{2n}(2k),$$

and
$$\int_0^\infty D_{2n+1}(x) \sin kx\, dx = (-1)^n \sqrt{\pi}\, D_{2n+1}(2k),$$

when n is any positive integer ... ... (B)

By applying the general result (A) to the integrals (B), we get

$$(-1)^n \sqrt{\alpha} \int_0^\infty \frac{D_{2n}(x)}{\cosh \frac{1}{2}\alpha x} dx = \sqrt{\beta} \int_0^\infty \frac{D_{2n}(x)}{\cosh \frac{1}{2} \beta x} dx, \quad (\alpha\beta = \pi)$$

$$(-1)^n \sqrt{\alpha} \int_0^\infty \frac{D_{2n}(x)}{1+2\cosh \alpha x} dx = \sqrt{\beta} \int_0^\infty \frac{D_{2n}(x)}{1+2\cosh \beta x} dx, \quad \left(\alpha\beta = \frac{\pi}{3}\right)$$

$$(-1)^n \sqrt{\alpha} \int_0^\infty \frac{D_{2n+1}(x)}{\sinh \frac{1}{2}\alpha x} dx = \sqrt{\beta} \int_0^\infty D_{2n+1}(x) \tanh \tfrac{1}{2}\beta x dx, \quad (\alpha\beta = \pi)$$

$$(-1)^n \sqrt{\alpha} \int_0^\infty D_{2n}(x) \frac{\cosh \frac{1}{2}\alpha x}{\cosh \alpha x} dx$$

$$= \sqrt{\beta} \int_0^\infty D_{2n}(x) \frac{\cosh \frac{1}{2}\beta x}{\cosh \beta x} dx; \quad \left(\alpha\beta = \frac{\pi}{2}\right) \quad \text{... (C)}$$

From the above results we get that for all *odd integral values of n*,

$$\int_0^\infty \frac{D_{2n}(x)}{\cosh \frac{1}{2} x \sqrt{\pi}} dx = 0,$$

$$\int_0^\infty \frac{D_{2n}(x)}{1+2\cosh x \sqrt{\pi/3}} dx = 0,$$

---

* *Proc. Edin. Math. Soc.*, XXXII (1914).

and $$\int_0^\infty \frac{D_{2n}(x)\cosh\sqrt{\pi}/2\sqrt{2}\,x}{\cosh x\sqrt{\pi/2}}\,dx = 0.$$

2. From (B), we have

$$\int_{-\infty}^{\infty} D_{2n}(k)\frac{dk}{k^m} = \frac{(-1)^n}{2^m\sqrt{\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_{2n}(x)\,\frac{\cos kx}{k^m}\,dx\,dk$$

$$= \frac{(-1^n)\sqrt{\pi}}{2^m\,\Gamma(m)\cos\frac{1}{2}m\pi}\int_{-\infty}^{\infty} D_{2n}(x)\frac{dx}{x^{1-m}},$$

in which m is positive and less than one.

If $m = \frac{1}{2}$, then for all *odd integral values of n*,

$$\int_0^\infty D_{2n}(x)\frac{dx}{\sqrt{x}} = 0.$$

Similarly

$$\int_0^\infty D_{2n+1}(x)\frac{dx}{\sqrt{x}} = 0. \qquad (n = \text{any positive odd integer}).$$

3. We have the formula

$$\int_0^\infty \frac{\sin kx}{e^{2\pi x}-1}\,dx = \frac{1}{2}\left(\frac{1}{e^k-1}+\frac{1}{2}-\frac{1}{k}\right).$$

Applying the result (A), we get

$$\sqrt{\alpha}\int_0^\infty \frac{D_{2n+1}(x)}{e^{\alpha x}-1}\,dx + (-1)^{n+1}\sqrt{\beta}\int_0^\infty \frac{D_{2n+1}(x)}{e^{\beta x}-1}$$

$$= \sqrt{\alpha}\int_0^\infty\left(\frac{1}{\alpha x}-\frac{1}{2}\right)D_{2n+1}(x)\,dx,$$

or $$(-1)^n\sqrt{\alpha}\int_0^\infty D_{2n+1}(x)\coth\tfrac{1}{2}\alpha x\,dx$$

$$= \sqrt{\beta}\int_0^\infty D_{2n+1}(x)\coth\tfrac{1}{2}\beta x\,dx. \qquad (\alpha\beta = \pi).$$

Let $\alpha=\beta=\sqrt{\pi}$ and $n = 2r+1$, where r is zero or any positive integer. Then we have

$$\int_0^\infty \frac{D_{4r+3}(x)}{e^{\sqrt{\pi}x}-1}dx=\frac{1}{2}\int_0^\infty\left(\frac{1}{\sqrt{\pi x}}-\frac{1}{2}\right)D_{4r+3}(x)\,dx$$

$$=\frac{1}{\sqrt{\pi}}\int_0^\infty D_{4r+3}(x)\frac{dx}{x}$$

$$=-\{D_{4r+2}(0)+(4r+2)D_{4r}(0)+(4r+2)4r\,D_{4r-2}(0)+\ldots$$
$$+(4r+2)4_r\ldots 2D_0(0)\},$$

where

$$D_{2r}(0)=\frac{(-1)^r(2r)!}{2^r(r)!},$$

$$=\frac{(4r+2!}{2^{2r+1}(2r+1)!}\;F\{1,-(2r+1);-(2r+\tfrac{1}{2});-1\}.$$

We easily deduce from the above

$$\int_0^\infty D_{4r+3}(x)\coth\tfrac{1}{2}\sqrt{\pi}x\,dx=0.$$

Since

$$\int_0^\infty \frac{\sin kx}{e^{2\pi x}+1}dx=\frac{1}{2k}-\frac{1}{2(e^{k/2}+1)}-\frac{1}{2(e^k-1)},$$

we get

$$(-1)^n\int_0^\infty\frac{D_{2n+1}(x)}{e^{\sqrt{\pi}x}+1}dx=\frac{1}{\sqrt{\pi}}\int_0^\infty D_{2n+1}(x)\frac{dx}{x}$$

$$-\int_0^\infty\frac{D_{2n+1}(x)}{e^{\frac{1}{2}\sqrt{\pi}x}+1}dx-\int_0^\infty\frac{D_{2n+1}(x)}{e^{\sqrt{\pi}x}-1}dx.$$

Whence

$$\int_0^\infty\frac{D_{4r+3}(x)}{e^{\sqrt{\pi}x}+1}dx=\int_0^\infty\frac{D_{4r+3}(x)}{e^{\frac{1}{2}\sqrt{\pi}x}+1}dx,$$

or $$\int_0^\infty D_{4r+3}(x)\tanh\tfrac{1}{2}\sqrt{\pi}x\,dx=\int_0^\infty D_{4r+3}(x)\tanh\tfrac{1}{4}\sqrt{\pi}x\,dx.$$

CORRESPONDENCE OF WAVE STATISTICS WITH CLASSICAL MECHANICS.

BY K. K. MUKHERJEE.

In a series of recent communications* in *Zeit. f. Phys.* K. C. Kar and the present writer have built up a theory of Wave Statistics giving a statistical interpretation of Schrödinger's Wave Mechanics. The object of the present paper is to establish the correspondence of our Wave Statistics with Classical Mechanics.

SECTION A.

Our Wave equation is

$$\frac{d^2D}{dt^2} = \frac{2\,(E-V)}{m} \Delta D \qquad \dots \quad (1)$$

where D denotes the density of the phase points and E, V, m have their usual significance. Now, it has been pointed out before that for a circular orbit

$$D \text{ varies as } e^{\frac{4\pi i E t}{h}} \qquad \dots \quad (2)$$

or

$$\dot{D} \quad ,, \quad \frac{4\pi i E}{h} D \qquad \dots \quad (3)$$

or

$$\bar{D} \quad ,, \quad \frac{4\pi i E}{h} \dot{D} \qquad \dots \quad (4)$$

The wave equation (1) may hence be written as

$$\frac{m}{2\,(E-V)} \cdot \frac{4\pi i E}{h} \dot{D} = \Delta D$$

or

$$\frac{2\pi i m}{h} \left(1 + \frac{V}{E}\right) \dot{D} = \Delta D$$

---

* Kar and Mukherjee *Zeit. f. Phys.* Bd. 59, S. 102. 1929; Bd. 60, S. 243, 1930; Bd. 56, S. 236, 1929; K. C. Kar. Bd. 61, S. 675, 1930; Bd. 64, S. 75, 1930.

or finally

$$\dot{\mathrm{D}} = \frac{h}{2\pi i m}\left(\Delta + \frac{8\pi^2 m}{h^2}\mathrm{V}\right)\mathrm{D} \qquad \ldots \quad (5)$$

It has been shown in the previous papers that the q-space is filled with stationary waves. The incident and reflected waves which combine to form these stationary waves contribute equally towards the density of the phase space. When, however, only one set of waves, either incident or reflected, is considered the value of the phase density will be half of that due to the stationary, waves and in that case the above equation (5) becomes

$$\dot{\mathrm{D}} = \frac{h}{4\pi i m}\left(\Delta + \frac{8\pi^2 m}{h^2}\mathrm{V}\right)\mathrm{D} \qquad \ldots \quad (6)$$

Now, if we denote by $\bar{q}$ the wave statistical average value of the space co-ordinate q, we have for only one set of waves

$$q = \tfrac{1}{4}\int q\,\mathrm{D}\bar{\mathrm{D}}\,d\tau \qquad \ldots \quad (7)$$

where $\bar{\mathrm{D}}$ is the conjugate of the compley function D, and D is normalised in such a way that

$$\int \frac{\mathrm{D}\bar{\mathrm{D}}}{4}\,d\tau = 1 \qquad \ldots \quad (8)$$

Again,

$$p = m\,\dot{q}$$

$$= \frac{m}{4}\int q\,(\dot{\mathrm{D}}\bar{\mathrm{D}} + \mathrm{D}\dot{\bar{\mathrm{D}}})\,d\tau, \text{ from (7) on differentiating}$$

$$= \frac{m}{4}\cdot\frac{h}{4\pi i m}\int q\,(\bar{\mathrm{D}}\Delta\mathrm{D} - \mathrm{D}\Delta\bar{\mathrm{D}})\,d\tau\,.\text{ from eq (6)}$$

$$= \frac{h}{16\,\pi i}\int q\,\mathrm{div}\,(\bar{\mathrm{D}}\,\mathrm{grad}\,\mathrm{D} - \mathrm{D}\,\mathrm{grad}\,\bar{\mathrm{D}})\,d\tau$$

$$= \frac{h}{16\pi i}\left[\int(\mathrm{D}\,\mathrm{grad}\,\bar{\mathrm{D}} - \bar{\mathrm{D}}\,\mathrm{grad}\,\mathrm{D})\,d\tau \right.$$

$$\left. + \int \mathrm{div}\,\{q\,(\bar{\mathrm{D}}\,\mathrm{grad}\,\mathrm{D} - \mathrm{D}\,\mathrm{grad}\,\bar{\mathrm{D}})\}\,d\tau\right].$$

Now, when the proper values (eigen values) of D are discrete, the surface integral becomes zero. Hence we have

$$p = \frac{h}{16\,\pi i} \int (\mathrm{D}\ \mathrm{grad}\ \bar{\mathrm{D}} - \bar{\mathrm{D}}\ \mathrm{grad}\ \mathrm{D})\ d\tau. \qquad \ldots \quad (9)$$

Again,

$$0 = \int \mathrm{grad}\ (\mathrm{D}\bar{\mathrm{D}})\ d\tau.$$

$$= \int (\mathrm{D}\ \mathrm{grad}\ \bar{\mathrm{D}} + \bar{\mathrm{D}}\ \mathrm{grad}\ \mathrm{D})\ d\tau. \qquad \ldots \quad (10)$$

Multiplying eq. (10) by $\frac{h}{16\,\pi i}$ and adding to eq. (9) we have

$$p = \frac{h}{8\pi i} \int \mathrm{D}\ \mathrm{grad}\ \bar{\mathrm{D}}\ d\tau \qquad \ldots \quad (11)$$

which is our expression for p, the wave statistical average of impulse.

We next proceed to establish a relation between the impulse moment P and the moment of force M.

In wave statictics we have, using eq. (11),

$$\mathrm{P} = \frac{h}{8\pi i} \int \mathrm{D}\ [r\ \mathrm{grad}\ \bar{\mathrm{D}}]\ d\tau.$$

On differentiating the above with respect to the time, we have

$$\dot{\mathrm{P}} = \frac{h}{8\pi i} \left\{ \int \dot{\mathrm{D}}\ [r\ \mathrm{grad}\ \bar{\mathrm{D}}]\ d\tau + \int \mathrm{D}\ [r\ \mathrm{grad}\ \dot{\bar{\mathrm{D}}}]\ d\tau \right\}$$

$$= \frac{h}{8\pi i} \left\{ \int \dot{\mathrm{D}}\ [r\ \mathrm{grad}\ \bar{\mathrm{D}}]\ d\tau - \int \dot{\bar{\mathrm{D}}}\ [r\ \mathrm{grad}\ \mathrm{D}]\ d\tau \right\}.$$

Substituting the values of $\dot{\mathrm{D}}$ and $\dot{\bar{\mathrm{D}}}$ from eq. (6) we get

$$\dot{\mathrm{P}} = -\frac{h^2}{32\pi^2 m} \left[\int \triangle \mathrm{D}\ [r\ \mathrm{grad}\ \bar{\mathrm{D}}]\ d\tau + \int \triangle\ \bar{\mathrm{D}}\ [r\ \mathrm{grad}\ \mathrm{D}]\ d\tau \right]$$

$$- \tfrac{1}{2} \left\{ \int \mathrm{V}\mathrm{D}[r\ \mathrm{grad}\ \bar{\mathrm{D}}]\ d\tau + \int \mathrm{V}\bar{\mathrm{D}}\ [r\ \mathrm{grad}\ \mathrm{D}]\ d\tau \right\} \quad \ldots \quad (12)$$

Now, the expression

$$\int \{ \triangle \mathrm{D}\ [r\ \mathrm{grad}\ \bar{\mathrm{D}}] + \triangle \bar{\mathrm{D}}\ [r\ \mathrm{grad}\ \mathrm{D}] \}\ d\tau$$

may be written as

$$\int [r,\ \Delta D \text{ grad } \bar{D} + \Delta \bar{D} \text{ grad } D]\, d\tau$$

which can be shown to be equal to zero, if we remember that ΔD grad $\bar{D} + \Delta \bar{D}$ grad D is the divergence of a certain symmetrical tensor.*

Again the expression

$$\int VD\,[r \text{ grad } \bar{D}]\, d\tau + \int V\bar{D}\,[r \text{ grad } D]\, d\tau$$

$$= -\int \bar{D}\,[r \text{ grad } (VD)]\, d\tau + \int V\bar{D}\,[r \text{ grad } D]\, d\tau$$

$$= \int D\bar{D}\,[r \text{ grad } V]\, d\tau.$$

Hence eq. (12) reduces to

$$\dot{P} = -\int \frac{D\bar{D}}{4}\,[r \text{ grad } V]\, d\tau = M. \qquad \dots \quad (13)$$

Thus it is proved that the relation $\dot{P} = M$ which is true in classical mechanics is true in wave statistics as well.

SECTION B.

In Sec. A, we have established in wave statistics the well-known relation ($\dot{P} = M$) of classical mechanics by a method which is essentially the same as that of Schrödinger. In this section we shall establish the same relation by an entirely different method far more simple than the previous one.

We have shewn in our previous papers that the velocity of the χ_1 waves in the q-space is

$$v = \sqrt{\frac{2\,(E - V)}{m}}. \qquad \dots \quad (14)$$

Hence, $$p = mv = \sqrt{2m\,(E - V)}. \qquad \dots \quad (15)$$

---

* Sommerfeld: *Wave Mechanics*, p. 247.

Now, we have from classical mechanics:

$$P = [r.p]$$

$$= [r.\sqrt{2m(E-V)}] \qquad \text{from eq. (15).}$$

On differentiating the above with respect to the time, we get

$$\dot{P} = \left[r.\frac{d}{dt}\left\{\sqrt{2m(E-V)}\right\}\right] \qquad \dots \quad (16)$$

$$= \left[r.\frac{d}{d(E-V)}\left\{\sqrt{2m(E-V)}\right\}\frac{d}{dx}(E-V).\frac{dx}{dt}\right]$$

$$= \left[r.\frac{m}{\sqrt{2m(E-V)}}.\sqrt{2m(E-V)}.\frac{d}{dx}(E-V)\right].$$

It is evident that for an electron moving in a field E is constant and thus we have from the above equation $\dot{P} = -[r \text{ grad } V]$.

And denoting the statistical average value of $\dot{P}$ by $\bar{\dot{P}}$, we have

$$\bar{\dot{P}} = \int \frac{\dot{P}D\bar{D}}{4}d\tau = -\int \frac{[r \text{ grad } V]D\bar{D}}{4}d\tau = M$$

Again, on differentiating eq. (15) with respect to the time, we get

$$\dot{p} = \frac{d}{dt}\sqrt{2m(E-V)} = -\text{grad } V. \qquad (\textit{vide} \text{ eq. (16)}).$$

Hence

$$\bar{\dot{p}} = \int \frac{\dot{p}D\bar{D}}{4}d\tau$$

$$= -\int \frac{\text{grad } V.\, D\bar{D}}{4}d\tau.$$

SECTION C.

In this section we shall find an expression for the current (S) in wave statistics.

From eq. (11) we have

$$p = mv = \frac{h}{8\pi i}\int \text{D grad } \overline{\text{D}}\, d\tau.$$

Hence, we have

$$v = \frac{h}{8\pi i m}\int \text{D grad } \overline{\text{D}}\, d\tau.$$

Again,

$$\text{S} = e \,.\, v = \frac{eh}{8\pi i m}\int \text{D grad } \overline{\text{D}}\, d\tau. \qquad \ldots \quad (17)$$

which is our expression for the current. Again, in the case of a circular orbit the solution of the wave equation in polar co-ordinates, gives

$$\text{D} = \text{CR P}^{n}_{e} \cos(\theta)\, e^{in\phi} \,.\, e^{\frac{4\pi i \text{E}}{h} t} \qquad \ldots \quad (18)$$

where C is a constant and R is a real function of r only. Hence we have

$$\overline{\text{D}} = \text{C . R P}^{n}_{e} \cos(\theta)\, e^{-in\phi} \,.\, e^{\frac{-4\pi i \text{E}}{h} t} \qquad \ldots \quad (19)$$

where $\overline{\text{D}}$ is the conjugate of D.

Since the current is a vector which has no gradient along r or θ, and because of the relations (18) and (19), eq. (17) becomes

$$\text{S} = \frac{eh}{8\pi i m} \,.\, \frac{-in\int \text{D}\overline{\text{D}}\, d\tau}{r \sin\theta} \qquad \ldots \quad (20)$$

$$= -\frac{enh}{2\pi m a}, \text{ (\textit{vide} eq. (8))}, \qquad \ldots \quad (21)$$

where a ($= r\sin\theta$) is the radius of the Bohr orbit of the electron.

Again, substituting in (21) the value of a $\left(= \frac{n^2 h^2}{4\pi^2 m e^2}\right)$ for the nth Bohr orbit, we have

$$\text{S} = -e \,.\, \frac{2\pi e^2}{nh} = -e \,.\, v \qquad \ldots \quad (22)$$

where v is the Bohr value of the velocity of the electron in the nth orbit. This proves conclusively that a moving electron can always be represented by a χ_1 — wave. We may remark here that Fermi's expression for the current can also be reduced to the form given above in eq. (21) According to Fermi*

$$S = \frac{-enh \mid \psi \mid^2}{2\pi mr \sin\theta} \quad \dots \quad (23)$$

where ψ is Schrödinger's wave function. Now, S in eq. (22) is not the total current as Fermi appears to think. Thus denoting the total current by J, we get from eq. (23) :

$$J = \int S d\tau = -\frac{enh}{2\pi m \; r\sin\theta} \int \mid \psi \mid^2 d\tau = -\frac{enh}{2\pi ma}$$

as in eq. (21).

We may add in conclusion that from the arguments put forward in proving eq. (22) it becomes evident that we can suppose the charge of a Bohr electron either to be spread over a volume in the q-space or to be concentrated at a point on the Bohr orbit as first assumed by Prof. Niels Bohr.

Finally I must thank Dr. K C. Kar for his valuable suggestions during the present investigation.

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* Fermi : *Nature*, December 18, 1926.

SYNTHETIC RELATIONS BETWEEN ANY THREE ELEMENTS OF A RIGHT-ANGLED TRIANGLE ON THE HYPERBOLIC PLANE.

By R. C. Bose,

Calcutta.

1. Introduction.—In Hyperbolic Geometry the angle of parallelism corresponding to a distance x is generally denoted by $\pi(x)$, while the distance of parallelism corresponding to the angle α is denoted by $\triangle(\alpha)$. The segment $x' = \triangle\left\{\frac{\pi}{2} - \pi(x)\right\}$ is said to be complementary to x. The functions π, $\triangle$, and the complementary function may be termed the fundamental functions of Hyperbolic Geometry. The relations between any *four* elements of a right-angled triangle in terms of the fundamental functions are well known[*]." The object of this short paper is to deduce more elegant relations between any *three* elements of a right-angled triangle in terms of the fundamental functions. The method of proof is purely synthetic no use having been made of Hyperbolic Trigonometry. In fact the relations here deduced by me may be regarded as the synthetic analogues of the trigonometrical relations between any three elements of a right-angled triangle.

2. I shall have occasion to use the following correspondence, between two bi-rectangular quadrilaterals, proved elsewhere by me.[†] (see fig. 1).

If $A_1B_1C_1D_1$ be a quadrilateral for which the angles at B_1 and C_1 are right, and the sides A_1D_1 and B_1C_1 cross, and if

$$A_1B_1 = b,\ \ B_1C_1 = p,\ \ C_1D_1 = d,\ \ D_1A_1 = q,$$
$$\angle B_1A_1D_1 = \lambda,\ \ \angle C_1D_1A_1 = \nu$$

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* Carslaw: *Non-Euclidean Geometry*. pp 59-60.

† R. C. Bose; 'The Theory of Associated Figures in Hyperbolic Geometry,' *Bull. Cal. Math. Soc.*, Vol. XIX (1928), p. 115.

then corresponding to it there will exist another quadrilateral $A_2B_2C_2D_2$, for which the angles at B_2 and C_2 are right, the sides A_2D_2 and B_2C_2 do not cross and

$$A_2B_2 = d,\ B_2C_2 = q',\ C_2D_2 = \triangle(\nu),\ D_2A_2 = p',$$
$$\angle B_2A_2D_2 = \pi - \pi(b),\ \angle C_2A_2D_2 = \lambda$$

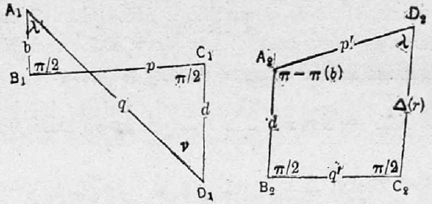

FIG. 1.

3. Let PQRST be a rectangular pentagon whose sides reckoned in order are x, y, z, u, v. S. Mukhopadhyaya has shown* that corresponding to this there will exist a triangle ABC right-angled at C for which

$$AB = y,\ BC = z',\ CA = x',\ \angle CAB = \pi(v),\ \angle ABC = \pi(u).$$

Now produce CB to C′ and AB to A′ making BC′ = CB and BA′ = AB. Join A′ and C′. Then corresponding to the crossed

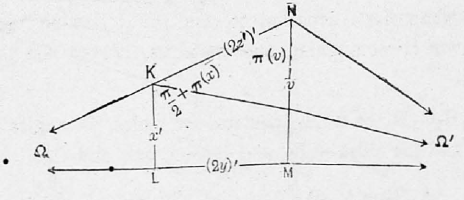

FIG. 2.

* S. Mukhopadhyaya: Geometrical Investigations on the correspondences between a right-angled triangle, a three right-angled quadrilateral and a rectangular pentagon in Hyperbolic Geometry.' *Bull. Cal. Math. Soc.*, Vol. XIII (1922-23), pp. 211—216.

bi-rectangular quadrilateral ACC′A′ we shall have (*cf.* para 2) a quadrilateral KLMN right-angled at L and M and for which (see Fig. 2)

$$\text{KL} = x', \;\; \text{LM} = (2y)', \;\; \text{MN} = v, \;\; \text{NL} = (2z')'.$$

$$\angle \text{LKN} = \frac{\pi}{2} + \pi\,(x), \;\; \angle\text{MNK} = \pi\,(v).$$

It is evident that NK is parallel to ML. Let Ω be the common point at infinity on these lines. Also draw $\text{K}\Omega'$ and $\text{N}\Omega'$ parallel to LM. Then in the infinite triangle $\text{KN}\Omega'$ we have

$$\angle\text{N K}\Omega' = 2\pi(x), \;\; \angle\text{K N}\Omega' = 2\pi(v), \;\; \text{KN} = (2z')',$$

whence

$$(2z')' = \Delta\,\{\,2\pi(x)\,\} + \Delta\,\{\,2\pi(v)\,\} \qquad \ldots \quad \text{(A)}$$

provided we follow Lobatschewsky in regarding the distance of parallelism corresponding to an obtuse angle as negative, and equal in magnitude to the distance of parallelism corresponding to its supplement.

The formula (A) gives a relation between any side of a rectangular pentagon, and the two opposite sides in terms of fundamental functions.

Next draw $\text{U}\Omega''$, $\text{V}\Omega''$, $\text{W}\Omega''$ perpendicular to KL, MN and LM at U, V, W respectively and parallel to KN, (see fig. 3) since

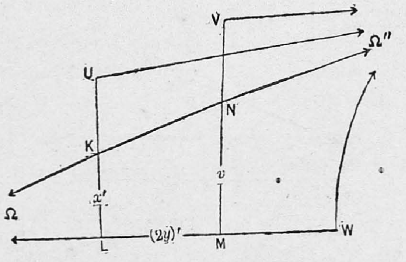

FIG. 3.

$\text{MV} = 2v$, $\text{MW} = (2v)'$ also $\text{UL} = 2x'$ and $\text{LM} = (2y)'$. Consequently

$$(2x')' = (2v)' + (2y)' \qquad \ldots \quad \text{(B)}$$

The formula (B) gives a relation between any side of a rectangular pentagon, and the two adjacent sides in terms of the fundamental functions.

4. Now let there be a right-angled triangle, with the hypotenuse c, sides a and b and angles λ and μ opposite to these sides respectively.

Set $\pi(a) = \alpha,\ \pi(b) = \beta,\ \pi(c) = \gamma,\ \Delta(\lambda) = l,\ \Delta(\mu) = m.$

Corresponding to this right-angled triangle there will exist a rectangular pentagon with sides a', c, b', 1, m reckoned in order. Applying the formulae (A) and (B) to this pentagon, we at once obtain for the right-angled triangle the following symmetrical relations in terms of the fundamental functions :—

$$
\begin{aligned}
(2c')' &= (2a')' + (2b')' = \Delta(2\lambda) + \Delta(2\mu) \\
(2a)' &= (2c)' + (2m)' = \Delta(2\lambda) - \Delta(2\beta) \\
(2b)' &= (2c)' + (2l)' = \Delta(2\mu) - \Delta(2\alpha) \qquad \ldots \quad \text{(C)} \\
(2l')' &= (2m)' + (2b')' = \Delta(2\gamma) - \Delta(2\alpha) \\
(2m')' &= (2l)' + (2a')' = \Delta(2\gamma) - \Delta(2\beta)
\end{aligned}
$$

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ON A MODIFICATION OF GIBBS'S STATISTICS FOR DISCONTINUOUS DISTRIBUTION IN PHASE-SPACE.

By B. N. Biswas and D. K. Sen.
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In the following paper we have been able, by replacing the continuous distribution in phase-space of the classical statistics by a sort of discontinuous distribution, to obtain the same sets of formulae as correspond to the classical, Bose-Einstein's and Fermi-Dirac's statistics. We have also given a modification of the latter (see equation (38)) which we believe will be applicable in all cases with proper values of λ.*

1. The canonical distribution put forward by Gibbs† is

$$P = e^{(\psi - U)/\Theta} \qquad \ldots \quad (1)$$

where P is equal to $\frac{\rho}{N}$, N being the total number of the systems which are represented in the phase-space and ρ the density of distribution; it appears from this that P has the dimension of inverse phase-volume, whereas the right-hand side of (1) has no dimension at all. This defect has been pointed out by Kar and Mazumdar‡ who have also suggested a modification for it We propose to use a modified form of equation (1) as will be free from this defect. It would appear later that our modification had the same expression as given by the authors just cited.

As the introduction of a constant factor on the right-hand side of equation (1) does not materially affect the probability co-efficient P,

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* On comparing this formula with that given by Brilloruin. (*viz.*, equation (26) on p. 147. *Selected Papers on Wave Mechanics* by De Broglie and Brilloruin, English Translation, 1929), one finds that although the two expressions are similar in form the constants involved in them have widely different meanings on account of the fundamental differences between our hypotheses and Brillo-ruins:

† Gibbs: *Elementary Principles of Statistical Mechanics.*

‡ K. C. Kar and R. C. Mazumdar: *Zeitschrift fur Physik*, *55*, 546, 1929.

regarded as a solution of the differential equation

$$\sum \left(\frac{\partial P}{\partial q_1}\cdot\frac{\partial U}{\partial p_1}-\frac{\partial P}{\partial p_1}\cdot\frac{\partial U}{\partial q_1}\right) = 0 \qquad \ldots \quad (2)$$

we write

$$P = \frac{1}{A}\, e^{(\psi - U)/\Theta} \qquad \ldots \quad (3)$$

where A is a constant having the dimension of phase-volume. The magnitude of ψ, but not its nature, may depend on the newly introduced constant. For, if we put

$$A = \lambda \,.\, A' = e^{\log \lambda}\,.\, A' \qquad \ldots \quad (4)$$

λ being an absolute constant and A' having the same dimension as A, the equation (3) takes the form

$$P = \frac{1}{A'}\,.\, e^{-\log \lambda\, \psi + (-U)/\Theta} \qquad \ldots \quad (5)$$

Let

$$\psi' = \psi - \Theta \log \lambda. \qquad \ldots \quad (6)$$

Then we have

$$P = \frac{1}{A'}\, e^{(\psi' - U)/\Theta} \qquad \ldots \quad (7)$$

which is of the same form as (3). It is clear, therefore, that ψ and A of (3) cannot both be regarded as independent quantities. We shall thus be justified in giving a suitable value to one of these constants and treating the other as the arbitrary constant of the solution of the differential equation satisfied by P.

2. The density of regions of phase-space as conceived by Gibbs and other writers is dependent on the systems being distributed in some continuous manner* in phase-space. It implies that the properties, such as positions, of the systems under consideration can differ from one another in a continuous manner. But it is not actually the case in the strict mathematical sense. To suit, therefore, the requirements of discontinuous variations which, however, would necessarily be small in magnitude, we assume that the representative points† in phase-space

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* Gibbs: *Elementary Principles of Statistical Mechanics.*

† We shall be using the notation R-point for a representative point, following the example of Rice. *Cf.* Rice's *Statistical Mechanics*, p. 237.

do not form a continuum but occupy distinct and discrete positions even though they may be quite close to each other.

With regard to such a discontinuous distribution, the density at a point ceases to have any meaning. For, let us select any point P in the space and take any small volume V surrounding that point; this volume, if not very small, will contain a number n of points. The density of this volume is given by n/V and the density at the point P is usually considered to be the limit to which this ratio tends when V tends to zero. Now P may happen to coincide with one of the R-points. We can then imagine a very small volume which would not contain any R-point other than P. Thus in the limit n/V has the value unity. Or, P might not have coincided with an R-point; it is easy to see that n/V tends to zero in this case. Hence the density at a point can only have the value 1 or 0 and thus can serve no purpose in our discussions. It seems therefore that we must not make V tend to absolute zero but to some suitably chosen small value which may be as small as we like but, nevertheless different from zero. This, we think, is really the implication when in defining density at a point, V is made to diminish indefinitely. Strictly speaking, this quantity should be termed density of a very small unit of volume instead of density at a point.*

Our small units or elements of phase-space will be such as could contain a large number of the discrete R-points, their number clearly depending on the magnitude of the element chosen. In other words, the size of our elementary phase-volume would be infinitesimally small the order of whose smallness is lower than that of the distance between two adjacent R-points. We may take $\rho . \Delta\tau$ as the number of R-points lying within an element of volume whose magnitude shall be denoted by $\Delta\tau$; ρ may be called the density of distribution. This quantity ρ would, in general, be different for different regions of phase-space but would practically be the same for any small region $\Delta\tau'$ within or in the immediate neighbourhood of $\Delta\tau$, provided that $\Delta\tau$ and $\Delta\tau'$ be of the same order of smallness. In fact, it is not possible for us to extend this definition of density to elements which would be small in magnitude in comparison with the element $\Delta\tau$ already chosen as our standard; such a smaller element will be provided by a 'cell' of

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* Cf. Jeans' *Dynamical Theory of Gases*, Ch. II.

Art. 4. It appears therefore that ρ will depend on the position of the element Δ_τ which is usually determined by the co-ordinates of a convenient point on it and the extensions of the element about this point. As we have assumed ρ to be independent of the extensions, provided they are of a certain definite order of smallness, we would be justified in regarding it as a function only of the co-ordinates of the point just mentioned. Further, for the sake of simplicity, this function may be assumed to be continuous although we have to consider its values at points which are isolated from one another. It is necessary to add here that ρ may depend on the time t, explicitly.

3. Let $p_1, p_2, \ldots p_n, q_1, \ldots q_n$ be the co-ordinates for the phase-space and let us consider an element of volume $\Delta\tau$ or $dp_1\, dp_2 \ldots dq_1 \ldots dq_n$. The number of R-points in this volume will therefore be $\rho\, dp_1 \ldots dp_n\, dq_1 \ldots dq_n$.

We want to find out how this number changes in an element of time dt. Accordingly let us consider how many R-points pass in or out of this volume on account of the change in p_1 in this interval. In the interval dt the R-points which enter the volume lie within the region between p_1 and $p_1 - \dot{p}_1\, dt$. Therefore their number is

$$\rho\, \dot{p}_1\, dt\, dp_1 \ldots dq_n.$$

Similarly the number which goes out of the volume will be given by

$$\rho\, \dot{p}_1\, dt\, dp_1 \ldots dq_n + \frac{d}{dp_1}\left(\rho\, \dot{p}_1\right) dp_1 \ldots dq_n\, dt,$$

because ρ and $\dot{p}_1$ have slightly different values at the two faces across which the R-points enter and pass out of this elementary volume. Now

$$\frac{d}{dp_1}\left(\rho\, \dot{p}_1\right) = \frac{d\rho}{dp_1}\, \dot{p}_1 + \rho\, \frac{d\dot{p}_1}{dp_1},$$

so that the total increase of number within the element due to the changes in all co-ordinates in the interval dt

$$= -\sum \frac{d\rho}{dp_1} \cdot \dot{p}_1 \cdot dp_1 \ldots dq_n \cdot dt - \sum \rho \cdot \frac{d\dot{p}_1}{dp_1} \cdot dp_1 \ldots dq_n\, dt \ldots \quad (8)$$

But by Hamilton's equations

$$\frac{d\dot{p}_r}{dp_r} + \frac{d\dot{q}_r}{dq_r} = 0, \qquad r = 1, 2, \ldots \ldots$$

Therefore the second sum in (8) vanishes and the total increase in number is given by the first sum which may be written

$$-\sum\left(\frac{d\rho}{dp_1}\cdot\dot{p}_1+\frac{d\rho}{dq_1}\cdot\dot{q}_1\right)dp_1\ldots dq_n\,.\,dt \qquad \ldots \quad (9)$$

To this should be added $\frac{d\rho}{dt}\,dp_1 \ldots\ldots dq_n\,dt$, if ρ contains t explicitly.

Hence for statistical equilibrium we must have

$$\sum\left(\frac{d\rho}{dp_1}\cdot\dot{p}_1+\frac{d\rho}{dq_1}\cdot\dot{q}_1\right)=0. \qquad \ldots \quad (10)$$

Thus we see that ρ as defined by us satisfies the same equation as the density distribution defined by Gibbs.

Now put $$P=\frac{\rho}{N} \qquad \ldots \quad (11)$$

where N represents the total number of systems represented in the phase-space. $\int P\,.\,d\tau$ taken over the whole phase-space

$$=\frac{1}{N}\int\rho\,.\,d\tau=\frac{1}{N}.N=1,$$

so that we may take P as the co-efficient of probability corresponding to the same region of phase-space as ρ.* It is easy to see that P would satisfy the same differential equation (10) as ρ.

This equation is obviously satisfied by P = a function of U, the energy, since $\dot{p}_r$, $\dot{q}_r$ can be expressed as partial derivatives of U. In view of the considerations put forth by Gibbs*, we propose to take the solution for P as given by (3).

When statistical equilibrium exists, the equation (10) is equivalent to Liouville's equation, $d\rho/dt = 0$. The above considerations show that this theorem is not affected provided we measure density with reference to volumes of order $\Delta\tau$. If, however, we pass on to smaller volumes (of higher order)—not to speak of 'at a point'—the theorem will not

* Gibbs: *Elementary Principles of Statistical Mechanics.*

hold. Since the distribution remains constant with time, it implies that there is no tendency of the R-points to congregate, as a whole, in special regions of the phase-space ; but if we consider a finite and large number of R-points, lying very close together within a small region of the phase-space, *e.g.*, Δ_T, they may or may not be uniformly distributed over this small region. It will be our endeavour in the next article to determine the nature of such distribution.

If S denote some average property of the ensemble of systems lying within such a small region, it would be the same over the whole range of phase-space. As the systems which could be represented by R-points situated within a very small volume, differ only slightly from each other, they may be considered as practically alike. S may then be regarded as representing the property of any single system to a first approximation. Hence, to this approximation, this property will be true for any domain of the phase-space ; in other words all points of phase-space are, to first approximation, equally likely. To a greater degree of accuracy, however, we must treat such small volumes Δ_T, and not points, as equally likely throughout the phase-space.

4. So far we have been discussing the probability P as connected with the quantity ρ. We shall now try to find out how the discrete R-points may be distributed in each element Δ_T. For this purpose we divide this element, and hence the whole phase-space, into a large number of smaller elements which we shall name 'cells.' These cells, although very small in magnitude,—being indeed of higher order of smallness than our previous element Δ_T,—have some definite magnitude. We shall, in this paper, treat the size of the cell to be the same and equal to ε, say. Some of these cells will contain no R-point, some will contain only one whilst some others contain two or more R-points.

Let a denote the number of cells, n the number of R-points inside the element Δ_T. Further, let p denote the probability that an R-point may lie in a particular cell. Then $(1-p)$ will be the probability that the point does not lie in that cell. It follows, therefore, from the usual discussions of probability that $ap^r(1-p)^{n-r}$ is equal to the number of cells containing r R-points, and $ap^{r+1}(1-p)^{n-r-1}$ to the number containing $(r+1)$ R-points. Therefore the ratio of the latter to the former

$$=\frac{p}{1-p}=\mu, \text{ say.}$$

That is, the number of cells containing $(r+1)$ R-points

$$= \mu \text{ times the number containing } r \text{ R-points.} \quad \dots \quad (12)$$

This can be seen to be true for $r = 0, 1, 2, \dots\dots$

Let a_0 = number of empty cells; then the numbers of cells with 1, 2, 3, ... R-points will be respectively

$$a_0 \mu, \ a_0 \mu^2, \ a_0 \mu^3, \ \dots \dots \qquad \dots \quad (13)$$

Now as n is limited, it is clear that no cell can contain an indefinitely large number of R-points. Let us suppose, therefore, that the maximum number included in a cell is r. Then the total number of cells in the volume Δ_T of phase-space, is

$$a = a_0 (1 + \mu + \mu^2 + \dots + \mu^r)$$

$$= \frac{a_0}{1-\mu}(1 - \mu^{r+1}) \qquad \dots \quad (14)$$

And the total number of R-points in this volume is

$$n = a_0 (\mu + 2 . \mu^2 + 3 . \mu^3 + \dots + r . \mu^r)$$

$$= a_0 \mu \left[\frac{1-\mu^r}{(1-\mu)^2} - \frac{r . \mu^r}{1-\mu} \right] . \qquad \dots \quad (15)$$

Therefore, from (14) and (15), we get

$$\frac{n}{a} = \frac{\mu}{1-\mu^{r+1}} \left[\frac{1 - \mu^r - r . \mu^r + r . \mu^{r+1}}{1-\mu} \right] \quad \dots \quad (16)$$

Now it should be borne in mind that (13) and the preceding formulae of probability are valid only when a_0 is large, so that we have to regard n and r, although they are pretty large numbers, as small in comparison with a. Thus a large number of cells remain empty, which means that μ is a small positive fraction. Therefore, μ^r and even $r . \mu^r$ and $r . \mu^{r+1}$ can be neglected as compared to μ; so as a close approximation, we get from (16)

$$\frac{n}{a} = \frac{\mu}{1-\mu} \text{ or } \mu = \frac{n}{a+n}. \qquad \dots \quad (17)$$

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NOTES AND QUESTIONS.

ASSOCIATE EDITORS:

M. BHIMASENA RAO; G. A. SRINIVASAN, M.A., L.T.

AND

N. SUNDARAM IYER, M.A.

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Notes and Questions.

India's Contribution to the Theory of Indeterminate Equations of the First Degree.

(*Continued from page 120*).

We have been able to deduce Āryabhaṭa's rule in indeterminate analysis on the supposition that the co-efficient of the unknown quantity on the divisor-side of a reduced equation like (7) becomes unity. Hence, if necessary, the last quotient should be so manipulated as to give unity as the co-efficient on the divisor-side. To make this point clear let us suppose that, after a number of reductions, the co-efficient on the dividend-side alone of a reduced equation becomes unity. Then the co-efficient on the divisor-side must be greater than unity ; and it is easy to see that the reduced equation will be of the form

$$r_{2n}\, x_{n+1} = r_{2n+1}\, y_n + \text{C}$$

where $$r_{2n+1} = 1,$$

i.e., $$r_{2n}\, x_{n+1} = y_n + \text{C}. \qquad \ldots \quad (15)$$

If the process of reducing the co-efficients be stopped here, Āryabhaṭa's rule *Śesaparasparabhaktam matigunam agrāntare ksiptam* (the last quotient of mutual division should be multiplied by any assumed integer and the product added to the difference of the given remainders) does not give the value of x_{n+1}. Therefore, we should continue the above process once more as follows :

Equation (15) may be written as

$$x_{n+1} = \{\, y_n - (r_{2n} - 1)\, x_{n+1} \,\} + \text{C}$$

or, $$x_{n+1} = y_{n+1} + \text{C}\,; \qquad \ldots \quad (16)$$

where $$y_{n+1} = y_n - (r_{2n} - 1)\, x_{n+1},$$

so that $$y_n = (r_{2n} - 1)\, x_{n+1} + y_{n+1},$$

$$x_{n+1} = 1 \,.\, y_{n+1} + \text{C}$$

$$y_{n+1} = t$$

$$\text{C} \equiv \text{C}$$

Now Āryabhaṭa's rule gives the values of x_{n+1} and y_n. Hence the values of x and y also can be found by the rule in such a case. It will appear from the above that when the co-efficient of the unknown quantity on the dividend-side alone becomes unity, the quotient of the next operation of division should be taken to be the co-efficient on the divisor-side *minus* unity so that the operation of division may be performed once more in order to give unity as the co-efficient on the divisor-side. According to Rangācārya such manipulation of a quotient is resorted to by Mahāvīra also. It will also be seen from the above that unity will be the co-efficient on the divisor-side or on the dividend-side according as the number of quotients of mutual division so as to leave no remainder is odd or even. Hence, to apply Āryabhaṭa's rule the number of such quotients must be made odd, by manipulation if necessary.

Āryabhaṭa's solution of the equation $Ax - By = C$ is the basis of the solutions suggested by his successors. The next Indian contributor to the theory of indeterminate equations of the first degree is Brahmagupta (b. 598 A. D.) who improves upon Āryabhaṭa's rule as explained below. We have seen that the required number in Āryabhaṭa's problem can be found even without equation (9). Or, in other words, to find the required number it is not necessary to use the quotient which occurs at the top of the column of quotients. Hence Brahmagupta rejects this quotient. He, therefore, says that the quotients of mutual division of the divisor corresponding to the smaller given remainder and the remainder left on dividing by it the divisor corresponding to the greater given remainder should be set down in a vertical line, one below another. This is expressed in the stanza :

Adhikāgrabhāgahārādūnāgracchedabhājitācchesam
Yat tat parasparahṛtam labdhamadho-dhaḥ pṛthak sthāpyam.

According to Brahmagupta it is not necessary to continue the operations of mutual division until the remainder becomes zero. We may stop at any stage if we can solve the resulting reduced equation. Thus we might stop at equation (6) if we could find, by inspection, an integer t such that, when the last remainder r_3 is multiplied by it, the product increased by C is divisible by r_2, *i e.*, if we could find, by inspection, a solution ($x_3 = s$, $y_1 = t$) of it. To find the value of y we

would then have to solve the equations

$$y = Q_2 x_1 + y_1$$
$$x_1 = Q_3 y_1 + x_2$$
$$y_1 = t$$
$$x_2 = s.$$

Or we might stop at equation (7) if we could find, by inspection, an integer t such that, when r_3 is multiplied by it, the product added to C is exactly divisible by the last remainder r_4, giving the quotient s. Therefore $y_2 = t$ and $x_2 = s$. In this case, to find y we have to solve the equations

$$y = Q_2 x_1 + y_1$$
$$x_1 = Q_3 y_1 + x_2$$
$$y_1 = Q_4 x_2 + y_2$$
$$x_2 = s$$
$$y_2 = t.$$

Brahmagupta accordingly directs that the remainder on the dividend-side should be multiplied by such an integer as will make the product, added to the difference between the given remainders, exactly divisible by the remainder on the divisor-side. His actual words are: *Śeṣaṃ tatheṣṭaguṇitaṃ yathāgrayorantareṇa saṃyuktaṃ śudhyati.*

It will appear from the above that in the column of quotients of mutual division the integer t (which Brahmagupta calls iṣṭa) should occupy the last place or the immediately preceding place according as we stop at remainder r_4 or at remainder r_3 or more generally according as we stop after obtaining an even or odd number of remainders (and, as a matter of course, of quotients). Brahmagupta says that beginning from the end (*i.e.*, from below) the multiplier t should be placed first and then the quotient s (*guṇakaḥ sthāpyo labdhañcāntyāt*)[18].

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18 This expression has been differently interpreted by the late Mahāmahopādhyāya Sudhākara Dvivedi who thinks that the word *antyāt* has no connection with the expression *guṇakaḥ sthāpyo labdhañca* and interprets the whole expression thus: "Below the quotients the multiplier t should be set down and then below it the quotient s; in performing the operations next suggested

The remaining part of Brahmagupta's rule is practically the same as that of Āryabhaṭa's rule. Following Āryabhaṭa we finally arrive at two numbers of which the upper number is a value of x and the lower is the corresponding value of y, whereas, following Brahmagupta, we finally arrive at two numbers of which the upper number is a value of y. Hence, according to Brahmagupta, the top-most number given by his column of quotients, etc., should be dealt with after Āryabhaṭa to obtain the least number answering to the two given divisors and to the two given remainders. Thus we have the following rule of Brahmagupta :

अधिकाग्रभागहारादूनाग्रच्छेद भाजिताच्छेषम् ।
यत् तत् परस्परहृतं लब्धमधोऽधः पृथक् स्थाप्यम् ।
शेषं तथेष्टगुणितं यथाऽग्रयोरन्तरेण संयुक्तम् ।
शुध्यति गुणकः स्थाप्योलब्धं चान्त्यादुपान्त्यगुणः ।
स्वोर्ध्वोऽन्त्ययुतोऽग्रान्तो हीनाग्रच्छेदभाजितः शेषम् ।
अधिकाग्रच्छेद हतमाधिकाग्रयुतं भवत्यग्रम् ।

Its meaning may be stated as follows:

"The quotients of mutual division of the divisor corresponding to the smaller given remainder and the remainder left on dividing by it the divisor corresponding to the greater given remainder should be placed, one below another, in the order in which they are obtained. After an even number of remainders have been obtained, the last remainder but one should be multiplied by such an integer as will make the product, added to the difference between the given remainders,

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in the rule one should begin from the end, *i.e.*, from below (*antyāt*)." This requires that we should stop the operations of mutual division after we have obtained an odd number of remainders ; and Dvivedī actually stops at such a stage in working out a numerical example. But in applying his rules for the solution of equations $Ax = By + C$ and $Ax = By - 1$, which are almost the same, Brahmagupta wants us to stop when an even number of remainders have been obtained. He says ; "*Evam samesu* (Proceed thus when the number of remainders and, therefore, of quotients, is even)." Again, the direction that the product of the last number but one and the number just above it should be added to the number just below it renders the use of the word *antyāt* superfluous if we accept Dvivedi's interpretation.

exactly divisible by the last remainder. The quotient thus obtained and the multiplier should be set down in order under the quotients of mutual division. The penultimate number in the column of quotients and multiplier should be multiplied by the number just above it and the product added to the last number which should then be rejected. This process should be continued. When the top-most number thus obtained, is divided by the divisor corresponding to the smaller given remainder the resulting remainder being multiplied by the divisor corresponding to the greater given remainder and added to the greater remainder, becomes the original number required."

The equation $Ax = By - 1$, on account of its importance in astronomical calculations, has received a separate treatment at the hands of Brahmagupta who has given it the name of *sthira-kuṭṭaka* (constant pulveriser). It could be solved by the above rule after reducing it to the form $By = Ax + 1$. But Brahmagupta has given a rule for its solution without this reduction. This rule is practically the same as the rule for the solution of the equation $Ax = By + C$. The difference is that we should here begin by dividing B by A instead of dividing A by B and that in finding a suitable integral multiplier after the operations of mutual division are stopped, we should subtract 1 from the product by the multiplier instead of adding it to the product.

The above rules require that the operations of mutual division should cease when an even number of remainders (and, therefore, of the quotients including the first quotient rejected) have been obtained. What will be the procedure if we have to stop the operations of mutual division after obtaining an odd number of remainders? Brahmagupta says that in such a case C and -1 should have their signs changed. This implies that C and -1 should be transposed to the divisor-side which would then be converted into the dividend-side. Now an integral multiplier will have to be found and the remaining operations carried on according to the above rules. Brahmagupta's own words are:

एवं समेषु विषमेष्वृणं धनं धनमृणं यदुक्तं तत् ।
ऋणधनयोर्व्यस्तत्वं गुण्यप्रक्षेपयोः कार्यम् ॥

In the equation $Ax = By + C$, B is also called the multiplicand (*guṇya*). Hence the meaning of the above verse may be stated thus:

"Proceed thus when the number of remainders is even. If the

number of remainders be odd, the negative should be made positive and the positive negative. This reversal of positive and negative signs should be made in respect of the positive and negative *kṣepas* to the given multiplicand, *i.e.*, in respect of the numbers to be added to or subtracted from the product of given multiplicand and the unknown multiplier to be found."

From *Brahma-sphuṭa-siddhānta* (Sudhākara Dvivedī's edition), Chapter XVIII, Verses 6 & 15, it appears that Brahmagupta could solve simultaneous indeterminate equations of the first degree.

We have seen that for the solution of equations $Ax = By + C$ and $Ax = By - 1$, as they are, according to Brahmagupta's rules one should begin by dividing B by A in both cases. But B is the divisor which corresponds to the greater remainder in the former equation and to the smaller remainder in the latter, as both the equations may result from attempts to solve problems of the type considered by Āryabhaṭa. Again, the *kṣepa* (the absolute term on the dividend-side) being positive in one case and negative in the other cannot be the difference between the given remainders in both the cases. Hence, the use of the expressions *adikāgrabhāgahāra* (divisor giving the greater remainder), *ūnāgrabhāgahāra* (divisor giving the smaller remainder) and *agrāntara* (difference between the given remainders) in the rule for the solution of the equation $Ax = By + C$ is responsible for a fresh rule for the solution of $Ax = By - 1$. One rule would have covered both the cases if A had been called the divisor (*cheda, hara,* or *hāra*) and B the dividend or multiplicand (*bhājya* or *guṇya*) and if C and -1 had been called *kṣepas* which may be positive or negative. This improvement has been effected by Mahāvīra (c. 850 A.D.) who gives the following rules for the solution of equation $Ax = By + C$ (where C is positive or negative) as well as for the solution of simultaneous indeterminate equations[19].

छित्वा छेदेन राशिं प्रथमफलमपोह्याप्तमन्योन्यभक्तं

स्थाप्योर्ध्वाधर्यतोऽधो मतिगुणमयुजेऽल्पे[20]ऽवशिष्टे धनर्णम् ॥

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[19] Because problems leading to simultaneous indeterminate equations follow these rules.

[20] अयुजेऽल्पे is more appropriate than अयुजाल्पे as अयुजे qualifies अवशिष्टे and not अल्पे.

छित्वाधः श्चोपरिघ्नोपरियुतहरभक्तो[21]ऽधिकाग्रस्य हारं
छित्वा छेदेन साग्रान्तरफलमधिकाग्रान्वितं हारघातम् ॥

The first three lines without the last two words *adhikāgrasyahāram* contain the rule for the solution of $Ax = By + C$. Here Mahāvīra finds the unknown quantity y only which is the *rās'i* or required number in a heap. For the sake of brevity Mahāvīra, like his predecessor Brahmagupta, uses the word for the unknown quantity to mean its co-efficient. The word *rās'i* in the above rule, therefore, means B, the co-efficient of the unknown quantity y on the dividend-side, *i.e.*, the given multiplier (of the unknown quantity y). Mahāvīra's rule for the solution of $Ax = By + C$ may then be translated thus:

" Dividing the given multiplier by the given divisor and rejecting the first quotient, place in order the other quotients of mutual division, one below another. Below these quotients should be placed the quotient obtained on dividing without a remainder by the smaller of the last two remainders the algebraic sum of the *kṣepa* and the other remainder multiplied by a suitable integer (*mati*). If the smaller of the last two remainders be odd in order of position, in which case it will be a multiplier and not a divisor, the sign of the *kṣepa* should be changed by transposition so as to convert the smaller of the last two remainders from a multiplier into a divisor. Place the *mati* under the last quotient. Now continue the operations of adding the lowest number to the product of the next two upper numbers and then rejecting the lowest number so treated. The number finally obtained in this manner, being divided by the divisor in the original equation, leaves a remainder which is the least value of the *rāśi* (the required multiplicand)."[22]

The fourth line together with the words *adhikāgrasyahāram* of the third line gives the rule for the solution of two or more simultaneous indeterminate equations. To interpret this rule let us solve the indeterminate equations

$$A_1x = B_1y + C_1$$
$$A_2z = B_2y + C_2.$$

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[21] हरभक्तः is more appropriate than हरभागः.

[22] This interpretation differs from that given by Rangācārya.

Here we may proceed in either of the following two ways :

(i) We may find general expressions for the values of y satisfying the two equations. In order that the two equations may be simultaneously satisfied, the two expressions must be equal. Thus, if β_1 and β_2 be any two values of y satisfying the above equations, $\beta_1 + A_1 i$ and $\beta_2 + A_2 t$ will be general expressions for the values of y satisfying the respective equations. If the same value of y satisfy both.

$$\beta_2 + A_2 t = \beta_1 + A_1 i$$

$$\therefore \quad A_2 t = A_1 i + \beta_1 - \beta_2 \text{ (supposing } \beta_1 > \beta_2\text{).}$$

Solving this indeterminate equation we find the value of i and then the value of $A_1 i + \beta_1$ which gives y.

(ii) We may find the general value $\beta_1 + A_1 i$ of y satisfying the first equation and then, by substitution in the second equation, find for what value of i, $\beta_1 + A_1 i$ will satisfy the second equation also. In this case we shall have to solve the indeterminate equation

$$A_2 z = A_1 B_2 i + B_2 \beta_1 + C_2.$$

We shall see hereafter that the second method is given by Bhāskara. The wording of Mahāvira's rule suggests that he gives the first method. This method depends on the solution of

$$A_2 t = A_1 i + \beta_1 - \beta_2.$$

Hence, by Mahāvīra's rule for the solution of such equations, we should begin by dividing A_1 (the divisor corresponding to the greater value β_1) by A_2 (the divisor corresponding to the smaller value β_2) and then proceed as directed by the rule. The value of i obtained by taking the difference between β_1 and β_2 as *kṣepa* should be multiplied by A_1 and then added to β_1 so as to give a value of y satisfying the two equations. It thus appears that by the word *agra* Mahāvīra means the values of y like β_1 and β_2, which satisfy the given equations. For, these values are the top-most (*agra*) numbers obtained from the respective columns of quotients and *matis*. For the sake of convenience β_1 and β_2 should be the least values of y.

Mahāvīra's rule for the solution of simultaneous indeterminate equations may now be interpreted as follows:

'Divide the divisor corresponding to the greater value of the unknown quantity by the other divisor and proceed as in the case of a single equation (This implies that the two equations should, first of all, be solved separately). The result finally obtained by taking as a *kṣepa* the difference between the two values of y satisfying the two equations, should be multiplied by the divisor corresponding to the greater value and then added to the greater value. The sum is a value of the unknown quantity occurring in the two equations."

The above rules of Mahāvīra, like Brahmagupta's rule explained above, require that a suitable multiplier (*mati* or *iṣṭi*) should be found by inspection. This may not be convenient in all cases. Hence Mahāvīra gives alternative rules not requiring the use of a *mati* other than unity. These rules are :

भाज्यच्छेदाग्रशेषैः प्रथमहृतिफलं त्याज्यमन्योन्यभक्तम् ।
न्यस्यान्ते साग्रमूर्ध्वैरुपरिगुणयुतं तैस्समानासमाने ॥
स्वर्णघ्नं व्याप्तहारौ गुणधनमृणयोश्चाधिकाग्रस्य हारम्
हृत्वा हृत्वा तु साग्रान्तरधनमधिकाग्रान्वितं हारघातम् ॥

Here, as in many other mathematical rules in Sanskrit verse, the grammar has been sacrificed for the sake of the metre. The word *antyam* would have been more appropriate than the word अन्ते *ante* in the second line. *Taissamānāsamāne* should have been corrected as *te samānāsamānah*. *Swarṇaghnau* would be better than *swarṇaghnam*. It is difficult to account for the dual form *ṛṇayoh* of the word *ṛṇa* while the word *dhana* occurs in the singular form as *guṇadhanam*. The two words *dhana* and *ṛṇa* ought to have been in the same case. The compound word *ṛṇadhanaguṇamāne* would have suited the metre and correctly expressed the meaning sought to be conveyed by the words *guṇadhanamṛṇayos'ca*. Then the third line would have stood thus :

स्वर्णघ्नौ व्याप्तहारावृणधनगुणमानेऽधिकाग्रस्य हारम् ।

Ante sāgram (= *antyam sāgram kāryam*) means that the last quotient of mutual division should be increased by the numerical value of the *agra* (by which term Mahāvīra here means *kṣepa* as shown by the second illustrative example). This implies that the *mati* (assumed multiplier) is 1 and that the corresponding divisor (the last remainder

used as a divisor) also is 1. *Swarṇaghnam* = *dhanarṇaguṇau*, *i.e.*, the required unknown multipliers of the given dividend corresponding to the positive and negative *kṣepas* or *agras*. *Vyāptahārau* = *vi* (*vigate*) *āpte* (*labdhaphale*) *yayoh tau vyāptau*; *tādṛs'au hārau*, *i.e.*, the divisor separately diminished by the results, that is, by the values of the unknown multiplier corresponding to the positive and negative *agras*.

The portion of the above verses containing Mahāvīra's rule for the value of y which satisfies the indeterminate equation

$$Ax = By + C$$

may now be translated thus :

" The first quotient of mutual division of the dividend, divisor, and their remainders, should be left out The other quotients being placed, one below another, in a vertical line, the last quotient should be increased by the numerical value of the given *agra*. Below this set down 1 as the *mati*. The lowermost number in the vertical chain thus obtained should be added to the product of the next two upper numbers and then rejected. The number finally obtained by repeating this last-mentioned operation is a value of the multiplier (y) of the dividend (B). The least value is obtained by taking from this value the divisor (A) as often as possible. This result corresponds to the positive or negative *agra* according as the number of remainders which are used as divisors is even or odd. When the reverse is the case, the least value of the required multiplier (y) is the given divisor diminished by the least value already obtained with the same *agra* with opposite sign."[23]

When the number of remainders which are used as divisors is even, this rule is practically the same as Āryabhaṭa's rule, the *mati* being taken as unity and the *kṣepa* being positive. When the number of such remainders is odd, the last remainder (which is 1 in this rule) occurs on the dividend-side which must be converted into the divisor-

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23 This interpretation is different from that given by Rangācārya. The fourth line together with the words अधिकाग्रस्य हारम् contains, as before, his rule for the solution of simultaneous equations.

side in order that the last remainder may be used as a divisor. This requires that the *kṣepa* or *agra* should be transposed from the dividend-side to the divisor-side. In order that the *kṣepa* or *agra* may be added arithmetically after transposition, its sign before transposition (*i.e.*, when it was on the dividend-side) must be negative. Thus the sign of the *kṣepa* or *agra* is to be positive or negative according as the number of remainders used as divisors is even or odd, *i.e*, according as the number of quotients *retained* is even or odd

Suppose that we can thus find $y = \beta$ so as to satisfy the equation $Ax = By + C$ but not the equation $Ax = By - C$. As this latter equation may be written as

$$A(B - x) = B(A - y) + C,$$

$y = A - \beta$ will satisfy this equation.

The use of the *kṣepa* in the column of quotients unnecessarily increases the product in each operation of adding the lowermost number in the column to the product of the next two upper numbers. Accordingly the rules given by Āryabhaṭa, Brahmagupta, and Mahāvīra require that, in most cases, the value of y obtained from the column of quotients should be divided by A in order to obtain the least value of y satisfying the equation. These rules first unnecessarily augment the value of y and then provide for its reduction. This defect has been removed by a mathematician of the tenth century who is known as the younger Āryabhaṭa (c. 950 A.D.) as distinguished from his senior namesake. He finds a solution (say, $x = \alpha, y = \beta$) of the equation $Ax = By \pm 1$ and then immediately deduces the solution ($x = C\alpha$, $y = C\beta$) of the equation $Ax = By \pm C$. If $C\alpha$ and $C\beta$ be respectively greater than B and A, the least values of x and y are found by taking away B and A respectively from $C\alpha$ and $C\beta$ the same largest number of times, as will be explained hereafter. The younger Āryabhaṭa's method is practically the same as that given by Bachet and Lagrange several centuries afterwards; but it was obtained in a different manner and does not rest on any result fortuitously obtained in the theory of continued fractions. On the other hand the well-known relation between the terms of two consecutive convergents of a vulgar fraction follows easily from the rule of the junior Āryabhaṭa.

Let us, therefore, describe this method fully. The younger Āryabhaṭa's rule is as follows:

भाज्यक्षेपच्छदा यथोदिताः संस्थिताः क-विधिरेषः ।
ते च करण्या भक्ता दृढाभिधाना अयं ख-विधिः ॥

भाज्यक्षेपौ ग-विधिः क्षेपच्छेदौ यदातदा घ-विधिः ।
भाज्यक्षेपौ क्षेपच्छदौ ङ-विधिर्विभिन्नकरणीभ्याम् ॥

एषां टा-शेषं स्याद्वल्लीकरणेऽत्र तैः सिद्धिः ।
ना-शेषं चेदिह तत् कुट्टाकारं न पृच्छको वेत्ति ॥

भाज्यहरावन्योन्यं विभजेत् टा-शेषकं भवेद् यावत् ।
सा वल्ली तेन हतेऽन्त्येनोर्ध्वे कान्विते स्फुटा वल्ली ॥

विषमसमत्वं ज्ञात्वाऽनष्टोपान्त्येन ताडिते स्वोर्ध्वे ।
स्वस्थानच्युतमन्त्यं योज्यमनेन प्रकारेण ॥

राशी कुट्टाख्यौ स्तो वक्ष्येऽन्यौ तौ सदा विषमजाख्यौ ।
सकृदेवच्छेदहृते भाज्ये शेषं यदा टा स्यात् ॥

लब्धं तदोर्ध्वकुट्टः शेषं चाधःस्थितो ज्ञेयः ।
कुट्टौ स्वक्षेपहतावूर्ध्वाधःस्थौ क्रमाद्भक्तौ ॥

निजभाज्यच्छेदाभ्यां फलगुणकौ शेषकौ भवतः ।
ग-विधावूर्ध्वं कुट्टं प्रश्नक्षेपेण संगुणयेत् ॥

करणीजक्षेपेणाऽधःस्थं घ-विधावतो व्यस्तम् ।
अनयोर्विध्योरेवं गुणितौ कुट्टौ क्रमाद्भक्तौ ॥

पृच्छककथितविभाज्यच्छेदाभ्यां फलगुणौ शेषौ ।
भाज्यक्षेपकरण्या ङ-विधावूर्ध्वं तलस्थमन्यकया ॥

हन्यान्मध्यफलगुणौ प्रश्नच्छेदं फलेन संगुणयेत् ।
भाज्यं गुणकेन तथा तद्विवरं हार इष्टः स्यात् ॥

प्रश्नक्षेपघ्नौ फलगुणकौ मध्यावभीष्टहारहृतौ ।
लब्धो प्रश्नविभाज्यच्छेदहृते फलगुणौ शेषौ ॥

It may be translated thus:

"*First operation*: Set down the dividend, *kṣepa* (additive or subtractive) and the divisor as stated in the question."

"*Second operation*: Make them prime to each other by dividing them by their greatest common divisor, if any."

"*Third operation*: Divide the dividend and *kṣepa* by their greatest common divisor, if any."

"*Fourth operation*: Similarly divide the *kṣepa* and divisor by their greatest common divisor, if any."

"*Fifth operation*: Divide the dividend and *kṣepa* by their G.C.D. and the *kṣepa* and divisor by their G.C.D. in case the two greatest common divisors are different."

"If the remainder obtained by mutual division of each of the above pairs be unity, they will lead to the formation of a vertical chain or column in the operation of *kuṭṭaka*; if the remainder be zero, the questioner does not know the method of *kuṭṭaka*."

"Divide the dividend by the divisor and then mutually divide their remainders until the remainder becomes unity. The quotients give rise to a vertical chain. This intermediate chain obtained by placing the quotients, one below another, becomes the corrected final chain when the penultimate quotient is replaced by unity added to the product of the penultimate quotient and the last quotient. Bearing in mind the oddness or evenness of the number of the quotients, the number just above the penultimate number in the chain should be multiplied by the penultimate number and then increased by the last number which should then be rejected. On repeating this process two numbers will be finally obtained which will be the values of the two unknown quantities. These two numbers also are called *kuṭṭas*. Let me now speak of two *kuṭṭas* which always result from an odd number of quotients. If, on dividing the dividend by the divisor, the first remainder becomes unity, the quotient is the upper *kuṭṭa* and the remainder (*i.e.*, unity) the lower *kuṭṭa*."[24]

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[24] This last statement should be compared with a similar statement made by Bachet (Dickson's *History of the Theory of Numbers*, Vol. II, p. 44).

"The upper and lower *kuṭṭas* thus obtained, being multiplied by the given *kṣepa* (additive or subtractive) and then respectively divided (as explained hereafter) by the given dividend and divisor, leave remainders which are respectively the quotient given by the given divisor and the required multiplier of the given dividend."

"If the third operation has been previously performed, the upper *kuṭṭa* should be multiplied by the original *kṣepa* of the question and the lower *kuṭṭa* by the *kṣepa* as reduced by division by the G. C. D. The reverse should be the case if the fourth operation has been performed. In the case of these two operations the *kuṭṭas*, after they have been multiplied as directed, should be divided respectively by the given dividend and divisor of the question as explained hereafter. The resulting remainders are respectively the quotient given by the divisor and the multiplier of the given dividend."

"If the fifth operation has been performed, the upper *kuṭṭa* should be multiplied by the G. C. D. of the dividend and *kṣepa* and the lower *kuṭṭa* by the other G. C. D. The products thus obtained will be *intermediate* (*i.e.*, not final) quotient and multiplier. Multiply the divisor and dividend of the question respectively by the *intermediate* quotient and multiplier and take the difference of the products as the divisor of the *intermediate* quotient and multiplier each multiplied by the *kṣepa* of the question. The quotients, divided respectively by the dividend and divisor of the question as explained hereafter, leave remainders which are respectively the required quotient and multiplier."

The younger Āryabhaṭa begins by removing the factors which are common to any pair of the given dividend, divisor, and *kṣepa*. The solution of the given equation is thus made to depend on the equation $Ax = By + C$ where no two of A, B, and C have any common factor.

SARADAKANTA GANGULI.

(*To be continued.*)

Solutions.

Question 1532.

(V. Ramaswamy Aiyar, M.A.):—Let the tangents at points P, Q of a given conic X intersect in T.

(*a*) Considering the in- and ex-centres of the triangle TPQ, show that if *one* of them lies on the curve, then *another* of them will also lie on the curve.

(*b*) Show that the locus of T in order that such may be the case is a confocal conic Y.

(*c*) Show that the relation between the conics X and Y, in this respect, is reciprocal.

(*d*) If X is a parabola, then Y is the confocal parabola having identical latus rectum.

Solution (1) *by T. R. Raghava Sastri*; (2) *by S. Narayanan, B.A., L T*

Also solved analytically by S. P. Ranganathachar and S. V. Minakshisundaram.

(1) (*a*) The equation to X is of the form $\alpha\beta = k\gamma^2$, where α, β, γ are the distances of any point on it from TP, TQ, and PQ respectively. If $(\alpha_1, \beta_1, \gamma_1)$ is on the conic, $(\alpha_1, \beta_1, -\gamma_1)$ is evidently on it. Hence (*a*),

(*b*) If T be the point (x_1, y_1) in cartesian co-ordinates and $X \equiv ax^2 + by^2 - 1 = 0$, the equation of the tangents from T is

$$\alpha\beta \equiv \frac{XX_1 - T^2}{\sqrt{[ab(y_1^2 - x_1^2) + b - a]^2 + 4a^2b^2x_1^2y_1^2}} = 0$$

(see Askwith's *Anal. Conics*, page 55, Q. 4),

where $\qquad X_1 \equiv ax_1^2 + by_1^2 - 1,$

and $\qquad T \equiv axx_1 + byy_1 - 1.$

The equation of PQ is $\gamma \equiv \dfrac{T}{\sqrt{a^2x_1^2 + b^2y_1^2}}$

So, putting the equation of the conic X in the form $\alpha\beta = k\gamma^2$, we see that if (1, 1, 1) is a point on the conic, $k=1$, the condition for which is, evidently

$$[ab(y_1^2 - x_1^2) + b - a]^2 + 4\, a^2 b^2 x_1^2 y_1^2 = (a^2 x_1^2 + b^2 y_1^2)^2$$

Transforming, simplifying and removing the trivial factor

$$ax_1^2 + by_1^2 - 1,$$

we get the locus of T in the form

$$ax^2 - by^2 = \frac{b-a}{b+a},$$

a confocal conic Y.

(*c*) On forming a similar locus for Y, in a similar manner, we come back to the original conic X.

(*d*) Taking $X \equiv y^2 - 4\,ax = 0$, and proceeding as above, the Y conic is found to be

$$y^2 = -\,4a\,(x - 2a).$$

(2) (*a*) Referred to TPQ, in trilinears the equation to the conics $\alpha^2 = k\beta\gamma$.

If (1, 1, 1) is on the conic, $k = 1$, and then (—1, 1, 1) is also on it. If (+ 1, — 1, 1) is on the conic, $k = -1$, and then (1, 1, —1) is also on it.

(*b*) [This part is given in Wolstenholme's *Mathematical Problems*, page 164, Question 1019.]

By the above reference, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be written in the form

$$\left\{\frac{x}{a}\cos(\alpha - \beta) + \frac{y}{b}\sin(\alpha - \beta) - 1\right\} \cdot \left\{\frac{x}{a}\cos\overline{\alpha + \beta} + \frac{y}{b}\sin\overline{\alpha + \beta} - 1\right\}$$

$$= \left\{\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - \cos\beta\right\}^2 \qquad \ldots \text{(A)}$$

where the left hand member equated to zero gives TP and TQ and the right hand member, the chord of contact PQ.

The in-centre of the Δ TPQ is given by

$$\frac{\frac{x}{a}\cos(\alpha-\beta)+\frac{y}{b}\sin(\alpha-\beta)-1}{\sqrt{\frac{\cos^2(\alpha-\beta)}{a^2}+\frac{\sin^2(\alpha-\beta)}{b^2}}}$$

$$=\frac{\frac{x}{a}\cos(\alpha+\beta)+\frac{y}{b}\sin(\alpha+\beta)-1}{\sqrt{\frac{\cos^2(\alpha+\beta)}{a^2}+\frac{\sin^2(\alpha+\beta)}{b^2}}}$$

$$=\frac{\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-\cos\beta}{\sqrt{\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}}}$$

Substituting these in (A), the condition that the in-centre may be on the conic is

$$\left(\frac{\cos^2(\alpha-\beta)}{a^2}+\frac{\sin^2(\alpha-\beta)}{b^2}\right)\left(\frac{\cos^2(\alpha+\beta)}{a^2}+\frac{\sin^2(\alpha+\beta)}{b^2}\right)$$

$$=\left(\frac{\cos^2\alpha}{a^2}+\frac{\sin^2\alpha}{b^2}\right)^2,$$

or, on simplification,

$$\frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\beta}=\frac{a^2-b^2}{a^2+b^2}. \qquad \dots \text{(B)}$$

The co-ordinates of T are given by

$$\left.\begin{aligned} x\cos\beta &= a\cos\alpha, \\ y\cos\beta &= b\sin\alpha. \end{aligned}\right\} \qquad \dots \text{(C)}$$

Eliminating α, β between (B) and (C), we obtain for the locus of T,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{a^2-b^2}{a^2+b^2} \qquad \dots \text{(D)}$$

(*c*) Let (D) be the given conic X; let

$$a'^2=\frac{a^2(a^2-b^2)}{a^2+b^2};\ b'^2=-\frac{b^2(a^2-b^2)}{a^2+b^2}$$

then (D) becomes $$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1.$$

The conic Y by (*b*) is $\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = \frac{a'^2 - b'^2}{a'^2 + b'^2}$

or when simplified $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

(*d*) Proceeding as in (*b*) the parabola $y^2 = 4ax$ may be written

$$4(my - m^2x - a)(ny - n^2x - a) = [y(m+n) - 2mnx - 2a]^2$$

where the left hand member gives TP, TQ and the right hand member PQ.

The condition for the in-centre of TPQ to be on this is seen as in (*b*) above to be

$$16m^2n^2(1+m^2)(1+n^2) = [(m+n)^2 + 4m^2n^2]^2 \quad \ldots \quad \text{(E)}$$

The co-ordinates of T are given by

$$xmn = a, \quad ymn = a(m+n) \quad \ldots \quad \text{(F)}$$

Eliminating m, n between (E) and (F) and simplifying, the locus of T is

$$(y^2 - 4ax)(y^2 + 4ax - 8a^2) = 0.$$

Omitting the trivial case $y^2 - 4ax = 0$, we have for Y the confocal parabola

$$y^2 + 4ax = 8a^2.$$

Question 1544.

(S. NARAYANAN, B.A., L.T.):—(i) If $p\dot{\alpha} + q\beta + r\dot{\gamma} = 0$ be an axis of a conic inscribed in the triangle of reference, show that its equation is

$$\sqrt{\alpha p(p^2 - q^2 - r^2 + 2qr\cos A)} + \sqrt{\beta q(q^2 - r^2 - p^2 + 2rp\cos B)}$$
$$+ \sqrt{\gamma r(r^2 - p^2 - q^2 + 2pq\cos C)} = 0$$

(ii) If the join of the foci of a conic inscribed in a triangle ABC subtend angles θ, ϕ, ψ at the vertices A, B, C, show that its equation is

$$\sin A \sqrt{\alpha/(\cos\theta - \cos A)} + \sin B \sqrt{\beta/(\cos\phi - \cos B)}$$
$$+ \sin C \sqrt{\gamma/(\cos\psi - \cos C)} = 0.$$

(iii) A conic being inscribed in the triangle ABC, if the joins of the vertices to a focus of the conic divide the angles A, B, C into parts α_1, α_2: β_1, β_2: γ_1, γ_2; show that its equation is

$$\sqrt{\alpha \sin A\ (\cot\alpha_1 + \cot\alpha_2)} + \sqrt{\beta \sin B\ (\cot\beta_1 + \cot\beta_2)}$$
$$+ \sqrt{\gamma \sin C\ (\cot\gamma_1 + \cot\gamma_2)} = 0.$$

Solution by S. Narayanan and V. A. Mahalingam.

(i) Let f, g, h be the trilinear co-ordinates of the focus of the in-conic. The tangential equation of the conic is

$$\left(lf + mg + nh\right)\left(\frac{l}{f} + \frac{m}{g} + \frac{n}{h}\right) = l^2 + m^2 + n^2 - 2mn\cos A$$
$$- 2nl\cos B - 2lm\cos C,$$

or, simplifying
$$\sum\left\{\frac{\frac{g}{h} + \frac{h}{g} + 2\cos A}{l}\right\} = 0.$$

The foci (f, g, h) and $\left(\frac{1}{f}, \frac{1}{g}, \frac{1}{h}\right)$ lie on $p\alpha + q\beta + r\gamma = 0$.

$$\therefore\ pf + qg + rh = 0$$

$$\frac{p}{f} + \frac{q}{g} + \frac{r}{h} = 0$$

Hence
$$\frac{g}{h} + \frac{h}{g} = \frac{p^2 - q^2 - r^2}{qr}.$$

Therefore the tangential equation of the in-conic is

$$\sum \frac{p^2 - q^2 - r^2 + 2qr\cos A}{lqr} = 0,$$

i.e.,
$$\sum \frac{p\,(p^2 - q^2 - r^2 + 2qr\cos A)}{l} = 0.$$

The point equation of the in-conic is

$$\sqrt{ap\,(p^2 - q^2 - r^2 + 2qr\cos A)} + \dots\dots = 0.$$

(ii) The line joining A to (f, g, h) is $\frac{\beta}{g} = \frac{\gamma}{h}$.

$$\therefore \quad \frac{\sin\left(\frac{A}{2} - \frac{\theta}{2}\right)}{\sin\left(\frac{A}{2} + \frac{\theta}{2}\right)} = \frac{g}{h} \text{ or } \tan\frac{\theta}{2} = \frac{h-g}{h+g}\tan\frac{A}{2}.$$

$$\therefore \quad \frac{h}{g} + \frac{g}{h} + 2\cos A = \frac{\tan\frac{A}{2} + \tan\frac{\theta}{2}}{\tan\frac{A}{2} - \tan\frac{\theta}{2}} + \frac{\tan\frac{A}{2} - \tan\frac{\theta}{2}}{\tan\frac{A}{2} + \tan\frac{\theta}{2}} + 2\cos A$$

$$= \frac{2\sin^2 A}{\cos\theta - \cos A}.$$

Substituting, we get the point equation of the conic

$$\sin A\sqrt{\frac{a}{(\cos\theta - \cos A)}} + \dots\dots = 0.$$

$$\text{(iii)} \quad \tan\alpha_1 = \tan\left(\frac{A}{2} - \frac{\theta}{2}\right) = \frac{\tan\frac{A}{2}\left[1 - \frac{h-g}{h+g}\right]}{1 + \frac{h-g}{h+g}\tan^2\frac{A}{2}} = \frac{g\sin A}{h + g\cos A}$$

similarly
$$\tan\alpha_2 = \frac{h\sin A}{g + h\cos A}.$$

$$\therefore \quad \frac{g}{h} + \frac{h}{g} + 2\cos A = \sin A\,(\cot\alpha_1 + \cot\alpha_2).$$

Hence the equation of the conic is

$$\sqrt{a\sin A\,(\cot\alpha_1 + \cot\alpha_2)} + \dots\dots = 0.$$

Question 1551.

(C N. SRINIVASIENGAR) :—(1) The parabolic curve of any skew ruled surface is the envelope of its non-rectilinear asymptotic lines.

(2) Every skew ruled surface (other than a quadric) possesses a finite number of plane lines of curvature given by the several generators comprising the parabolic curve.

Solution and Remarks by the Proposer.

I have proved previously (*vide* Qn. 1536) that the parabolic curve of a skew ruled surface consists of a number of isolated generators. The above are further properties of this parabolic curve.

[If L, M, N denote the fundamental magnitudes of the second order in curvilinear co-ordinates, the equation of the asymptotic lines is given by $Ldp^2 + 2\,Mdpdq + Ndq^2 = 0$. The parabolic curve is given by $LN - M^2 = 0$, which is the discriminant for dq/dp of this differential equation. It is well-known that for a general surface, the parabolic curve represents the cuspidal locus of the system of asymptotic lines. The ruled surface presents an exceptional case].

Any surface must possess two systems of asymptotic lines, and through any ordinary point of the surface there must pass one asymptotic line of each system. At a parabolic point, there is only one inflexional tangent. Hence, if we consider a point on one of the generators comprising the parabolic curve, one of the asymptotic lines through the point is the generator itself, while the other which must be a curve, must touch this generator, since this is the single inflexional tangent at the point. Thus, all the curved asymptotic lines of the surface touch the parabolic curve. This proves the result (1) above.

Since the parabolic curve consists of straight lines, the two inflexional directions at any point on one of these lines coincide with the direction of the straight line. Therefore, one of the principal directions also coincides with this. Hence, *the parabolic curve of any skew ruled surface is always a line of curvature*. Result (2) above expresses this fact in other words. The reason for the change of wordings is on account of the importance which has been attached to the study of

plane lines of curvature on surfaces (*vide* Forsyth's *Differential Geometry*, Chapter IX).

It will be interesting to account for the above theorems analytically. Consider the equations of a ruled surface in the form

$$x = f_1(p) + q\,\phi_1(p)$$
$$y = f_2(p) + q\,\phi_2(p)$$
$$z = f_3(p) + q\,\phi_3(p)$$

whose generators are given by p = constant.

It is found that $$N = 0,$$

$$M\sqrt{EG - F^2} = \begin{vmatrix} f_1' & f_2' & f_3' \\ \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \end{vmatrix},$$

$$L\sqrt{EG - F^2} = \begin{vmatrix} f_1' & f_2' & f_3' \\ \phi_1 & \phi_2 & \phi_3 \\ f_1'' & f_2'' & f_3'' \end{vmatrix} + q \begin{vmatrix} f_1' & f_2' & f_3' \\ \phi_1 & \phi_2 & \phi_3 \\ \phi_1'' & \phi_2'' & \phi_3'' \end{vmatrix}.$$

The equation of the parabolic curve reduces to $M = 0$.

The equation of the asymptotic lines becomes

$$L dp^2 + 2M dp dq = 0.$$

This gives p = constant—the generators, and the equation

$$L dp + 2M dq = 0.$$

This equation therefore represents the non-rectilinear asymptotic lines. But if $p=k$ be a root of the equation $M = 0$, $p = k$ satisfies the differential equation $L dp + 2M dq = 0$. The envelope property may be inferred from the fact that the values of $\frac{dq}{dp}$ are equal (to infinity) for $p = k$ and for the curve $L dp + 2M dq = 0$ at a point of intersection.

The equation $$\begin{vmatrix} E dp + F dq & F dp + G dq \\ L dp + M dq & M dp + N dq \end{vmatrix} = 0$$

giving the lines of curvature is clearly satisfied by $M = 0$, since M is a function of p only, and N is identically zero. This proves result (2).

Question 1587.

(K. J. SANJANA) :—Find the complete solution of the equation

$$(2x + 1)y'' + (4x - 2)y' - 8y = (6x^2 + x - 3)e^x$$

it being given that $y = e^{mx}$ is a solution when the right-hand side is zero and find the constant m.

Solutions by B. Ramamurti, S. Narayanan, Hansraj Gupta and M. S. Srinivasachari.

On putting $y = e^{mx}$ in the complementary equation m is easily found to be -2

To find another solution of the same equation let us try

$$y = Ax^2 + Bx + C.$$

Then

$$(2x + 1)\,2A + (4x - 2)(2Ax + B) - 8(Ax^2 + Bx + C) \equiv 0.$$

This gives $\qquad B = 0,\ C = \dfrac{A}{4}.$

Hence $\quad A(x^2 + \frac{1}{4})$ is a solution where A is an arbitrary constant.

It remains only to find a particular solution of the given equation For this put $y = e^x . v$. The equation becomes

$$(2x + 1)\frac{d^2v}{dx^2} + 8x\frac{dv}{dx} + (6x - 9)\,v = 6x^2 + x - 3.$$

Let us try $v = Ax + B$.

This gives $A = 1$, $B = \frac{1}{3}$ or $v = x + \frac{1}{3}$.

Hence the complete solution of the given equation is

$$Ae^{-2x} + B(x^2 + \tfrac{1}{4}) + e^x(x + \tfrac{1}{3}).$$

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Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1602. (KAVASJI J. SANJANA AND KAIKOBAD F. VAKIL) :— Two sides of a triangle are $19 \times 23 \times 59$ and $19 \times 23 \times 83$, and the included angle is

$$\cos^{-1}\left\{\frac{132858283}{682234187}\right\}:$$

show that the third side is rational and find it.

1603. (A. A KRISHNASWAMI AYYANGAR) :—

If $S + \lambda_1 S'$, $S + \lambda_2 S'$, $S + \lambda_3 S'$ be the line-pairs of a four-point system of conics determined by S, S', prove that the three conics $S - \lambda_1 S'$, $S - \lambda_2 S'$, $S - \lambda_3 S'$ have double contact with the ϕ-conic of S, S'.

Hence deduce a property of trinodal quartics.

1604. (B. RAMAMURTI) :—A variable line intersects a pair of opposite edges of a tetrahedron inscribed in a quadric Q. Show that the locus of the double points of the involution on the line determined by the pairs of points of intersection with the pair of opposite edges and the quadric, is a quadric containing the other 4 edges and doubly apolar to Q.

1605. (K. J. SANJANA AND K. F. VAKIL) :—In connection with Mr. R. Ramanujan's Question No 1598 (*Journal*, Aug. 1931), prove the following two results :—

(1) The length of the side BC of the pentagon ABCDE, required to be constructed, is given by the equation

$$BC \, . \, DG = 16992 \times (3 \, . \, 5 \, . \, 7 \, . \, 13 \, . \, 47 \, . \, 59 \, . \, 61 \, . \, 83 \, . \, 317)^{\frac{1}{2}}$$

Note that the length of DG is given in the Question.

(2) If P and Q are the lengths of the perpendiculars drawn from the point G respectively upon CD and BF, then

$$P : Q = 2867 : 437.$$

LIST OF JOURNALS & BOOKS RECEIVED IN THE LIBRARY

during the months of October and November 1931.

1 Acta Mathematica, **57**, 1, 2.

2 American Journal of Mathematics, **53**, 4.

3 American Mathematical Monthly, **38**, 7, 8.

4 Annales de l'ecole Normale Superieure, September and October, 1931.

5 Annals of Mathematics, **32**, 4.

6 Astrophysical Journal, **74**, 2.

7 Bulletin of the American Mathematical Society, **37**, 8, 9.

8 Bulletin des Sciences Mathematiques, **55**, July, Aug., Sep. and Oct. 1931.

9 Indian Physico-Mathematical Journal, **2**, 2.

10 Jahresbericht der deutschen mathematiker Vereinigung, **41**, 1, 2, 3, 4.

11 Japanese Journal of Mathematics, **8**, 2.

12 Journal of the Faculty of Science, **1**, 1, 2.

13 Mathematical Gazette, **5**, 215 (3 copies).

14 Mathematische Annalen, **105**, 3, 4.

15 Monthly Notices of the Royal Astronomical Society, **91**, 9.

16 Philosophical Magazine, **12**, 79.

17 Philosophical Transactions of the Royal Society, **230**, 221 to 322.

18 Proceedings of the Cambridge Philosophical Society, **27**, 4.

19 Proceedings of the London Mathematical Society, **32**, 7 (3 copies).

20 Proceedings of the Physico-Mathematical Society, of Japan, **13**, 7.

21 Proceedings of the Royal Society of London, **133**, 822.

22 Quarterly Journal of Mathematics, **2**, 7.

23 Tohoku Mathematical Journal, **34**, 2.

Books.

1 University of Madras, The Calendar for 1931-32, Vol. 1, Parts 1 & 2.

2 Bombay Unniversity Calendar, 1928-1929; 1929-1930.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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